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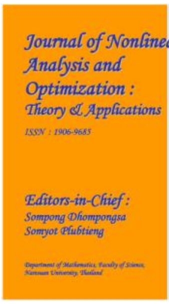

COVERING COVER EDGE PEBBLING NUMBER OF WHEEL GRAPH AND APPLICATION OF COVER EDGE PEBBLING NUMBER IN CRYPTOGRAPHY

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ABSTRACT
The Cover edge pebbling number of a graph G is defined to be the minimum number of pebbles such that however the pebbles are initially placed on the edges of the graph it should allow at least one pebble to reach all the edges of the graph. It is denoted by $CP_E(G)$.The Covering cover edge pebbling number of a graph G is the minimum number of pebbles required to cover the set of edges which forms the covering of the given Graph. It is denoted by $C_CP_E(G)$. Cryptography is the study of Coding for the purpose of securing information from third parties. It involves the terms of encryption and Decryption which involves the process of converting Plaintext to Ciphertext and the Ciphertext to plaintext. In this paper the covering cover edge pebbling number of Wheel graphs and application of the cover edge pebbling number in cryptography is discussed.
Keywords: Wheel Graph, Covering cover edge pebbling number, Cover edge pebbling number, Cryptography, Encryption, Decryption.

1. INTRODUCTION
Graph pebbling is a branch of graph theory where the pebbles are placed on the edges or on the vertices of the graph and the 'Game play' is composed of a series of pebbling moves.
Initially Lagarias and Saks first suggested the game of pebbling and it was introduced by Chung [1], into the literature. An edge pebble move is the process of removal of two pebbles from one edge and addition of a pebble to its adjacent edge. Crull introduced the concept of cover pebbling [2] and Priscilla Paul A introduced the concepts of Edge pebbling move, edge pebbling number, cover edge pebbling number and covering cover edge pebbling number [3]. The edge pebbling number, cover edge pebbling number of some standard graphs like path, complete graph, friendship graph, star graph and some families of graphs like Helm graph, crown graph and pan graph are determined in [4][5].
The covering cover edge pebbling number of friendship graph, odd path and even path are determined in [6].

2. COVERING COVER EDGE PEBBLING NUMBER
Definition 2.1[6]: The covering cover edge pebbling number of a graph G denoted by $C_cP_E(G)$ is defined as the minimum number of pebbles required to cover the edges that forms a covering for G in all possible configurations of pebbles.
Theorem 2.1:
$C_CP_E(W_4) = 5$
Proof:
Let the edge covering set be {e*, e′} such that e* is any one of the key edge and e′ is an edge which is at a maximum distance from e*.
Case (i):
If the pebbles are placed on the key edges adjacent to both e*, e′ only four pebbles are required to cover the edges in the edge covering set.

Case (ii):

If the pebbles on e* to cover the edges in the edge covering set. Four pebbles are to be placed on e* to reach e′ and one more pebble to cover e*. Totally five pebbles are required to cover e*, e′.

Case (iii):

If the pebbles are placed on the key edges not in the edge covering set, four pebbles are required to cover the edges in the edge covering set.

Therefore, **$C_C P_E(W_4) = 5$.**

Theorem 2.2:

$C_C P_E(W_5) = 9$.

Proof:

Let the edge covering set be {e*, e_{11}, e_1*} such that e* is any one of the key edge and e_1* is an edge which is at a maximum distance from e* and the edge e_{11} is incident with the common vertex and e*.

Case (i):

If the pebbles are placed on e* then, four pebbles are to be placed on it to cover e_1* and two more pebbles are needed to cover e_{11}. Moreover, one more pebble is to be placed on it to cover itself. So, totally seven pebbles are required to cover the edges in the edge covering set.

Case (ii):

Now place the pebbles on e_1* in the edge covering set. Eight pebbles are to be placed on e_1* to reach e_{11} and e*. Moreover, one more pebble is to be placed on it to cover itself. So, totally nine pebbles are required to cover the edges in the edge covering set and if the pebbles are placed on e_{11}, seven pebbles are required to cover all the edges in the edge covering set.

Case (iii):

Now place the pebbles on the internal edges not in the edge covering set. A maximum of six pebbles are required to cover the edges in the edge covering set.

Case (iv):

Now place the pebbles on the key edges not in the edge covering set. There are two edges in which the pebbles are to be placed. In those two edges the possible configurations of pebbles are (1, 6), (2, 5), (3, 4) and vice-versa. In all configuration the edges in the edge covering set can be covered. So, totally seven pebbles are required to cover the edges in the edge covering set.

Therefore, **$C_C P_E(W_5) = 9$.**

Theorem 2.3:

$C_C P_E(W_6) = 10$

Proof:

Let the edge covering set be {e*, e_{11}, e_1*} such that e* is any one of the key edge and e_1* is an edge which is at a maximum distance from e* and the edge e_{11} is incident with the common vertex and e*.

Case(i):

If the pebbles are placed on e*only eight pebbles are required to cover the edges in the edge covering set.

Case(ii):

If the pebbles are placed on e_1* and e_{11} only seven pebbles are required to cover the edges in the edge covering set.

Case(iii):

If the pebbles are placed on the internal edges not in the covering set only eight pebbles are required to cover the edges in the edge covering set.

Case(iv):

If the pebbles are placed on the key edges not in the edge covering set , a maximum of ten pebbles are required to cover the edges in the edge covering set.

Therefore, **$C_C P_E(W_6) = 10$**

Theorem 2.4:

$C_C P_E(W_7) = 16$

Proof:

Let the edge covering set be {e*, e_1*, e_2*, e_{11}} such that it contains a key edge e*, the key edges which are at distance one from e* say e_1* and e_2* and the edge that is incident with the common vertex and e* say e_{11}.

Case(i):

If the pebbles are placed on e* only eleven pebbles are required to cover the edges in the edge covering set.

Case(ii):

If the pebbles are placed on and e_{11}, e_1*, e_2* thirteen pebbles are required to cover the edges in the edge covering set.

Case(iii):

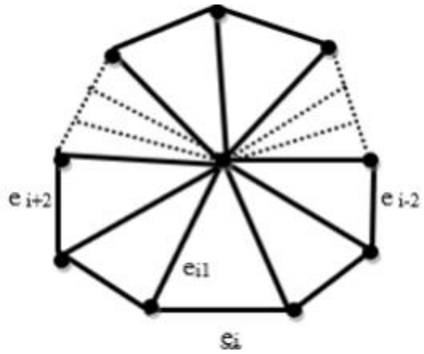
If the pebbles are placed on the internal edges not in the covering set, a maximum of 12 pebbles are required to cover the edges in the edge covering set.

Case(iv):

If the pebbles are placed on the key edges not in the edge covering set, a maximum of sixteen pebbles are required. (i.e) Placing pebbles on the edges adjacent with e_1* sixteen pebbles are required to cover the edges in the covering set and with e_2* fourteen pebbles are required to cover the edges in the edge covering set.

Therefore, **$C_CP_E(W_7) = 16$.**

Theorem 2.5:



$C_CP_E(W_{2n+1}) = 8n - 8$ for $n \geq 3$ and $C_CP_E(W_{2n}) = 8n - 15$ for $n \geq 4$

Proof:

Let the key edges be named as $e_1, e_2, \ldots, e_i, \ldots, e_{2n+1}$ and also consider the edge e_{i1} that is adjacent with all the internal edges, e_i, e_{i+1} and incident with the common vertex.

For Odd Wheel graphs (W_{2n+1} , $n \geq 3$)

Let the edge covering set be **S = {** $e_i, e_{i+2}, \ldots, e_{i-2}, e_{i1}$**}**

Case(i):

If the pebbles are placed on e_i, 8n -13 pebbles are required to cover the edges in the edge covering set.

Case(ii):

If the pebbles are placed on $e_j \in S$, ($j \neq i, i1$) and e_{i1} a maximum of 4n -1 pebbles are required to cover the edges in the edge covering set.

Case(iii):

If the pebbles are placed on the internal edges not in the edge covering set, a maximum of 8n -12 pebbles are required to cover the edges in the edge covering set.

Case(iv):

If the pebbles are placed on the key edges not in the edge covering set, a maximum of 8n -8 pebbles are required to cover the edges in the edge covering set. Therefore,

$C_CP_E(W_{2n+1}) = 8n - 8$.

For Even Wheel graphs (W_{2n}, $n \geq 4$)

Let the edge covering set be **S' =** { $e_i, e_{i+2}, \ldots, e_{i-1}, e_{i1}$**}**

Case(i):

If the pebbles are placed on e_i, 8n -15 pebbles are required to cover the edges in the edge covering set.

Case(ii):

If the pebbles are placed on $e_j \in S'$, ($j \neq i, i1$), 8n -19 pebbles are required to cover the edges in the edge covering set and if the pebbles are placed on e_{i1}, a maximum of 4n -3 pebbles are required to cover the edges in the edge covering set

Case(iii):

If the pebbles are placed on the internal edges not in the edge covering set, a maximum of 4(n -1) pebbles are required to cover the edges in the edge covering set.

Case(iv):

If the pebbles are placed on the key edges not in the edge covering set, a maximum of 8(n-2) pebbles are required to cover the edges in the edge covering set.

Therefore, $\mathbf{C_CP_E(W_{2n}) = 8n - 15}$.

3. APPLICATION OF COVER EDGE PEBBLING NUMBER IN CRYPTOGRAPHY

Definition 3.1 [8]: Cryptography is the study of methods of sending messages in disguised form so that only the intended recipients can remove the disguise and read the message.

Definition 3.2 [8]: The message we want to send is called the plaintext and the disguised message is called the ciphertext.

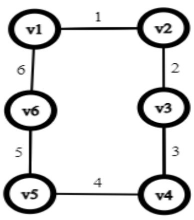
Definition 3.3 [8]: The process of converting a plaintext to ciphertext is called enciphering or encryption and the reverse process is called deciphering or decryption.

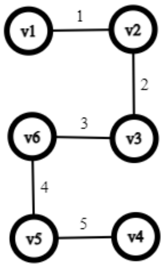
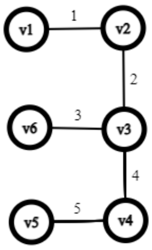
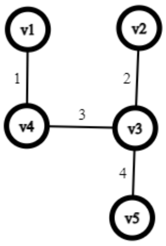
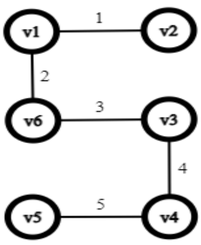
Definition 3.4 [8]: An Enciphering transformation is a function that takes any plaintext message unit and gives us a ciphertext message unit. In other words, it is a map f from the set $\wp$ of all possible plaintext message units to the set C of all possible ciphertext message units. We shall always assume that f is a 1-to-1 correspondence. That is, given a ciphertext message unit ,there is one and only one plaintext message unit for which it is the encryption. The Deciphering transformation is the map f^{-1} which goes back and recovers the plaintext from the ciphertext. $i.e., \wp \xrightarrow{f} C \xrightarrow{f^{-1}} \wp$

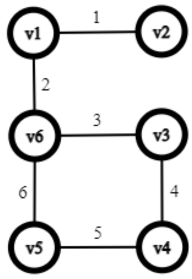
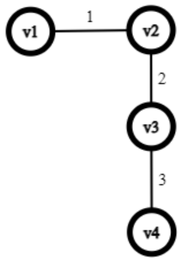
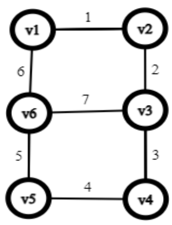
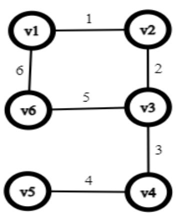
Definition 3.5[3]: A cover edge pebbling number $CP_E(G)$ of a graph G is defined as, however the pebbles are initially placed on the edges, the minimum number of pebbles required to place a pebble in all edges.

APPLICATION:

Consider the numerical numbers from 0 to 9 as digital numerals.

Number (n)	**Graphical Representation**	**Key Edges**	$\mathbf{CP_E(n)}$
0		All the edges are key edges	21
1		All the edges are key edges	3

2		1 and 5 are the key edges	31
3		1 and 5 are the key edges	19
4		1 is the key edge	11
5		1 and 5 are the key edges	31

6		1 is the key edge	27
7		1 and 3 are the key edges	7
8		1 and 4 are the key edges	25
9		4 is the key edge	27

Encryption Decryption table:

n	$\mathbf{CP_E(n)}$	$\mathbf{CP_E(n)} + \lfloor 13n/40 \rfloor$	**P′**
0	21	21+0	21
1	3	3+0	03
2	31	31+0	31
3	19	19+0	19
4	11	11+1	12
5	31	31+1	32

6	27	27+1	28
7	7	7+2	09
8	25	25+2	27
9	27	27+2	29

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